Stochastic perturbations in vortex-tube dynamics

L. Moriconi[1] and F. A. S. Nobre[2,3]
[1]*Instituto de Física, Universidade Federal do Rio de Janeiro, Caixa Postal 68528, 21945-970, Rio de Janeiro, RJ, Brazil*
[2]*Centro Brasileiro de Pesquisas Físicas, Rua Xavier Sigaud, 150, Rio de Janeiro, RJ 22290-180, Brazil*
[3]*Universidade Regional do Cariri, Rua Cel. Antônio Luis, 1160, Crato, CE 63100-000, Brazil*
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A dual lattice vortex formulation of homogeneous turbulence is developed, within the Martin-Siggia-Rose field theoretical approach. It consists of a generalization of the usual dipole version of the Navier-Stokes equations, known to hold in the limit of vanishing external forcing. We investigate, as a straightforward application of our formalism, the dynamics of closed vortex tubes, randomly stirred at large length scales by Gaussian stochastic forces. We find that besides the usual self-induced propagation, the vortex tube evolution may be effectively modeled through the introduction of an additional white-noise correlated velocity field background. The resulting phenomenological picture is closely related to observations previously reported from a wavelet decomposition analysis of turbulent flow configurations.

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I. INTRODUCTION

Considerable progress has been achieved in the past two decades concerning the kinematics of turbulent coherent structures, a fact intimately associated to the improving performance of computer and experimental resources [1]. However, the relevant dynamical properties of the evolution and interaction of the energy-containing eddies—believed by many to comprise the key for a fundamental understanding of intermittency and other turbulence characteristics—are still essentially unknown. As a concrete illustration of the present theoretical limitations, it is worth recalling the difficulties faced in the study of wall turbulence. Even though the main flow patterns have been identified in that situation [2,3], there is, for instance, no solid theoretical foundation for the logarithmic law of the wall.

An ideal arena for the investigation of the dynamical and kinematical issues is provided by homogeneous isotropic turbulence. Direct numerical simulations have showed clearly that at moderately high Reynolds numbers, the flow is dominated by long-lived vortex tubes with small cross-sectional dimensions (defined around the Kolmogorov dissipation length) and sizes extending up to the integral scale [4–8]. It is also known, as set on a firm ground by Farge *et al.* [9], through wavelet decomposition methods, that most of the turbulent kinetic energy is carried by the vortex tubes, which are surrounded on their turn by a background incoherent flow.

Several analytical studies have addressed over the years the picture of homogeneous turbulent flows in terms of vortex tubes, either from the dynamical or kinematical viewpoints (see Ref. [10] for a comprehensive review). Among the former, growing attention has been devoted to Lundgren's model [11], based on the evolution of strained spiral vortices, which are transformed into tubelike structures and are probably generated in real flows through shear layer instabilities [8,12]. In contrast, in the kinematical approach, the dynamical details are bypassed, and an effective account of the statistical stationary regime of the "vortex tube gas" is attempted, as in the works of Chorin [15], where a connection with standard polymer statistical mechanics is drawn, and Hatakeyama and Kambe [16], whose focus relies on the properties of flow configurations related to multifractal distributions of vortex filaments (modeled as Burgers vortices).

Our initial aim in this paper is to establish, in Sec. II, an alternative formulation of the turbulence problem, incorporating into the usual stochastic approach [17] the physical insight suggested from experimental and numerical investigations, which, as commented on above, place vorticity coherent structures on a central stage. More specifically, we will implement, with the help of the Martin-Siggia-Rose functional formalism [18], an exact statistical lattice vortex description of the flow's dynamics [13–15], which contains, as a special case, the known dipole form of the Navier-Stokes equations. Next, in Sec. III having in mind modeling matters (and, thus, nonrigorous arguments), we use the lattice vortex formalism just developed to advance a phenomenological scheme describing the evolution of vortex tubes forced at large scales by stochastic forces. In particular, we also consider the effective force-force correlation function employed in the renormalization-group analysis of turbulence [19–22], decaying in Fourier space as k^{-3}. In Sec. IV, we find that the stochastic perturbations due to the random external forcing may be effectively interpreted as resulting from the vortex tube advection by a white-noise correlated velocity background flow. We determine the one-dimensional energy spectrum $E(k) \sim k^2$ of the background flow, including its dependence upon the energy transfer rate, and the integral and viscous scales as well. It is interesting to note that a "thermal-like" energy spectrum, superimposed to the Kolmogorov one, was indeed observed in the numerical wavelet analysis of turbulent configurations performed in Ref. [9]. To conclude, in Sec. V, we summarize and discuss our main results.

II. DUAL LATTICE VORTEX FORMULATION

As largely known, a systematic approach to the statistical description of homogeneous isotropic turbulence, which is concerned with the flow's small-scale properties, is yielded by the stochastic generalization of the Navier-Stokes equations [17],

$$\partial_t v_\alpha + v_\beta \partial_\beta v_\alpha = \nu \partial^2 v_\alpha - \partial_\alpha P + f_\alpha,$$

$$\partial_\alpha v_\alpha = 0. \tag{2.1}$$

Above, $f_\alpha = f_\alpha(\vec{x},t)$ denotes a Gaussian random force, defined at some large length scale L, with vanishing expectation value and the two-point correlation function

$$\langle f_\alpha(\vec{x},t) f_\beta(\vec{x}',t')\rangle = \delta_{\alpha\beta} F(|\vec{x}-\vec{x}'|)\delta(t-t'). \tag{2.2}$$

It follows from Novikov's theorem [23] that energy is injected at large scales with pumping rate $\mathcal{E} = \langle f_\alpha v_\alpha\rangle = 3F(0)/_2 \equiv 3D_0/_2$. Furthermore, according to the standard Kolmogorov phenomenology [24], it is conjectured that dissipation takes place around the microscopic scale given by $\eta \sim D_0^{-1/4}\nu^{3/4}$, where viscous effects become relevant. The Reynolds number, depending only on the extreme scales L and η, is $R_e \sim (L/\eta)^{4/3}$.

Let the spatial part of the force-force correlator be written as

$$F(|\vec{x}-\vec{x}'|) = \frac{D_0 m}{\pi^2}\int d^3\vec{k}\frac{\exp(i\vec{k}\cdot\vec{x})}{(k^2+m^2)^2} = D_0 \exp(-m|\vec{x}-\vec{x}'|), \tag{2.3}$$

with $m \equiv 1/L$. An important feature of expression (2.3), which does not necessarily hold for other admissible choices of the force-force correlation function, is that its inverse has a simple local form. Actually, we get, from Eq. (2.3),

$$F^{-1}(|\vec{x}-\vec{x}'|) = \frac{1}{8\pi D_0 m}(\partial^2 - m^2)^2 \delta^3(\vec{x}-\vec{x}'). \tag{2.4}$$

Notwithstanding the fact that the Fourier transform of $F(|\vec{x}-\vec{x}'|)$ is regarded in principle as a regularized version of Dirac's δ function in some appropriate functional space [25], we will discuss, later on, dynamical effects related to the alternative definition

$$F(|\vec{x}-\vec{x}'|) = \frac{D_0}{4\pi}\int d^3\vec{k} k^{-3} \exp[i\vec{k}\cdot(\vec{x}-\vec{x}')], \tag{2.5}$$

which has been a crucial ingredient in the renormalization-group studies of turbulence [19–22]. In Eq. (2.5), the integration in Fourier space is bound to the region $1/L < k < 1/\eta$. Observe that in this case, the mean energy input rate per octave is fixed to $D_0 \ln 2$, and we have $F(0) = D_0 \ln(L/\eta)$.

Considering the bulk of experimental and numerical evidence that favors the picture of turbulence as a vortex tube gas, from now on our attention will be focused on the vorticity dynamics implied by the stochastic Navier-Stokes equations. It is convenient, thus, to work with the stochastic Helmholtz equation, straightforwardly derived from Eqs. (2.1) as

$$\partial_t \omega_\alpha + v_\beta \partial_\beta \omega_\alpha - \omega_\beta \partial_\beta v_\alpha = \nu \partial^2 \omega_\alpha + f^\star_\alpha, \tag{2.6}$$

where $f^\star_\alpha = \epsilon_{\alpha\beta\gamma}\partial_\beta f_\gamma$ and $\omega_\alpha = \epsilon_{\alpha\beta\gamma}\partial_\beta v_\gamma$ is the vorticity field. We obtain, using Eq. (2.2), the correlator

$$\begin{aligned}\langle f^\star_\alpha(\vec{x},t) f^\star_\beta(\vec{x}',t')\rangle &\equiv D_{\alpha\beta}(|\vec{x}-\vec{x}'|)\delta(t-t') \\ &= \epsilon_{\alpha\rho\sigma}\epsilon_{\beta\gamma\eta}\partial_\rho\partial'_\gamma\langle f_\sigma(\vec{x},t) f_\eta(\vec{x}',t')\rangle \\ &= (\partial_\alpha\partial_\beta - \delta_{\alpha\beta}\partial^2)F(|\vec{x}-\vec{x}'|)\delta(t-t').\end{aligned} \tag{2.7}$$

We state now, in field theoretical language, what is meant by the stochastic evolution problem. Defining $\omega^0_\alpha(\vec{x})$ as the vorticity field at a certain time instant t_0, we are interested in finding the probability density functional $Z = Z[\omega_\alpha(\vec{x}), t_1 | \omega^0_\alpha(\vec{x}), t_0]$ for the observation of vorticity $\omega_\alpha(\vec{x})$ at a latter time instant t_1. Within the path-integral version of the Martin-Siggia-Rose formalism [18], it follows, from Eqs. (2.6) and (2.7), that

$$Z = \mathcal{N}\int D\hat{\omega}_\alpha D\omega_\alpha \exp(iS), \tag{2.8}$$

where $\mathcal{N}$ is a normalization constant [26], to assure that $\int D\omega_\alpha(\vec{x}) Z[\omega_\alpha(\vec{x}), t_1 | \omega^0_\alpha(\vec{x}), t_0] = 1$, and

$$S = \int_{t_0}^{t_1} dt \left\{ \int d^3\vec{x}\,\hat{\omega}_\alpha(\partial_t\omega_\alpha + v_\beta\partial_\beta\omega_\alpha - \omega_\beta\partial_\beta v_\alpha - \nu\partial^2\omega_\alpha) + i\int d^3\vec{x} d^3\vec{x}'\,\hat{\omega}_\alpha(\vec{x},t) D_{\alpha\beta}(|\vec{x}-\vec{x}'|)\hat{\omega}_\beta(\vec{x}',t) \right\}. \tag{2.9}$$

An effective model of turbulent dynamics would be naturally attained if the velocity and vorticity fields that appear in Eq. (2.9) were expressed as a sum over the contributions produced exclusively by relevant flow profiles. The basic difficulty here regards the selection and parametrization of such configurations. A promising starting point is to establish a set of "building blocks" that could be used to represent the usually observed coherent structures. We recall that vortex sheets or tubes, in particular, can be exactly obtained in a simple way as linear combinations of elementary closed vorticity rings (or "vorticity plaquettes"), through a lattice vortex construction [14,15] originally devised in the realm of superfluid physics [13]. Just define a cubic lattice, with spacing parameter $\epsilon \to 0$ (i.e., much smaller than the Kolmogorov dissipation length η), whose sites are written as $\vec{x}_p = \epsilon(p_1\hat{x}_1 + p_2\hat{x}_2 + p_3\hat{x}_3)$, where the p_i's are integers. The vector position $\vec{x}_p$ is taken to be the common vertex of three plaquettes oriented according to the unit vectors $\hat{x}_\sigma$, as shown in Fig. 1. In a self-evident notation, an arbitrary plaquette is completely characterized by the vector doublet $\mathcal{P} = (\vec{x}_p, \hat{x}_\sigma)$. Furthermore, by definition, the plaquette's boundary $\partial\mathcal{P}$ is identified to a line vortex (vortex tube with vanishing cross section) which carries vorticity flux $\phi_\sigma(\vec{x}_p, t)$. Of course, a square line vortex is an ill-defined mathematical object, due to the divergence of the velocity field on its corners [27]. However, as a simple regularization procedure, we impose an ultraviolet cutoff $\Lambda \equiv 1/\epsilon$ in the Fourier-transformed kernel of the operator that maps vorticity into velocity. In rough terms, this is equiva-

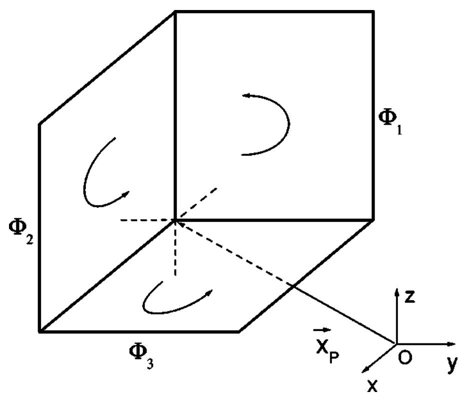

FIG. 1. The three oriented plaquettes which have the common reference position $\vec{x}_p$, and carry, on their boundaries, vorticity fluxes Φ_1, Φ_2, and Φ_3.

lent to replacing the line vortex by a vortex tube with a cross section of radius $\sim\epsilon$. It is important to remark that the lattice vortex is in fact an "overcomplete" basis for the description of general flow configurations. In other words, a given velocity configuration can be expressed in several different (and equivalent) ways as a superposition of the elementary vorticity plaquettes. This representation freedom is intimately related to the existence of a gauge field structure hidden in the stochastic model, as will be discussed below.

Considering the plaquette $\mathcal{P}=(\vec{x}_p,\hat{x}_\sigma)$, let $x_\alpha(s)\in\partial\mathcal{P}$ be a point parametrized by the arclength $0\leqslant s\leqslant 4\epsilon$ of the oriented boundary path that starts at the reference point $\vec{x}_p$ and ends at $x_\alpha(s)$. The vorticity field associated with this plaquette is

$$\omega_\alpha=\phi_\sigma(\vec{x}_p,t)\delta(n_1)\delta(n_2)\frac{d}{ds}x_\alpha(s), \tag{2.10}$$

where n_1 and n_2 are the coordinates along the normal and binormal directions on the line vortex (the binormal vector is defined as $\hat{x}_\sigma$) carrying vorticity flux $\phi_\sigma(\vec{x}_p,t)$. The central idea underlying the lattice vortex representation is, then, to substitute Eq. (2.10) in the Martin-Siggia-Rose action (2.9) and perform afterwards the sum over all the plaquettes. Introducing $F_{\alpha\beta}\equiv\partial_\alpha\hat{\omega}_\beta-\partial_\beta\hat{\omega}_\alpha$, we get, for a single plaquette, the following relations:

$$\int d^3\vec{x}\,\hat{\omega}_\alpha\partial_t\omega_\alpha=\partial_t\phi_\sigma(\vec{x}_p,t)\oint_{\partial\mathcal{P}}dx_\alpha\hat{\omega}_\alpha$$

$$=-\partial_t\phi_\sigma(\vec{x}_p,t)\oint_{\partial\mathcal{P}}dx_\alpha\partial^{-2}\partial_\beta F_{\alpha\beta},$$

$$\int d^3\vec{x}\,\hat{\omega}_\alpha(v_\beta\partial_\beta\omega_\alpha-\omega_\beta\partial_\beta v_\alpha)=\phi_\sigma(\vec{x}_p,t)\oint_{\partial\mathcal{P}}dx_\alpha F_{\alpha\beta}v_\beta,$$

$$\int d^3\vec{x}\,\hat{\omega}_\alpha\partial^2\omega_\alpha=\phi_\sigma(\vec{x}_p,t)\oint_{\partial\mathcal{P}}dx_\alpha\partial^2\hat{\omega}_\alpha$$

$$=-\phi_\sigma(\vec{x}_p,t)\oint_{\partial\mathcal{P}}dx_\alpha\partial_\beta F_{\alpha\beta}. \tag{2.11}$$

Thus, the Martin-Siggia-Rose action becomes

$$S=\int_{t_0}^{t_1}dt\Bigg\{\sum_{\mathcal{P}}\oint_{\partial\mathcal{P}}dx_\alpha[\partial_t\phi_\sigma\partial^{-2}\partial_\beta F_{\alpha\beta}+\phi_\sigma F_{\alpha\beta}v_\beta+\nu\phi_\sigma\partial_\beta F_{\alpha\beta}]+\frac{i}{2}\int d^3\vec{x}d^3\vec{x}'F_{\alpha\beta}(\vec{x},t)F(|\vec{x}-\vec{x}'|)F_{\alpha\beta}(\vec{x}',t)\Bigg\}. \tag{2.12}$$

It is worth noting that the action (2.12) is invariant under the local transformation $\hat{\omega}_\alpha\rightarrow\hat{\omega}_\alpha+\partial_\alpha\chi$. In fact, it turns out that the transition probability for small time intervals, derived from Eq. (2.8), with Eq. (2.12), is well approximated by the expectation value of a product of loop operators, computed in a nonlocal, three-dimensional, $U(1)$ gauge theory.

Drawing upon the gauge field theory correspondence, we define now the dual field strength,

$$\hat{\phi}_\alpha(\vec{x},t)\equiv\tfrac{1}{2}\epsilon_{\alpha\beta\gamma}F_{\beta\gamma}(\vec{x},t), \tag{2.13}$$

which satisfies $\partial_\alpha\hat{\phi}_\alpha=0$ and, additionally,

$$F_{\alpha\beta}=\epsilon_{\alpha\beta\gamma}\hat{\phi}_\gamma,$$

$$\tfrac{1}{2}F_{\alpha\beta}F_{\alpha\beta}=\hat{\phi}_\alpha\hat{\phi}_\alpha,$$

$$dx_\alpha F_{\alpha\beta}v_\beta=\epsilon_{\alpha\beta\gamma}dx_\alpha v_\beta\hat{\phi}_\gamma. \tag{2.14}$$

We find, substituting the above relations in Eq. (2.12),

$$S=-\int_{t_0}^{t_1}dt\Bigg\{\sum_{\mathcal{P}}\oint_{\partial\mathcal{P}}dx_\alpha\epsilon_{\alpha\beta\gamma}[\partial_t\phi_\sigma(\vec{x}_p,t)\partial^{-2}\partial_\beta\hat{\phi}_\gamma+\phi_\sigma(\vec{x}_p,t)\hat{\phi}_\beta v_\gamma-\nu\phi_\sigma(\vec{x}_p,t)\partial_\beta\hat{\phi}_\gamma]+i\int d^3\vec{x}d^3\vec{x}'\,\hat{\phi}_\alpha(\vec{x},t)F(|\vec{x}-\vec{x}'|)\hat{\phi}_\alpha(\vec{x}',t)\Bigg\}. \tag{2.15}$$

The solenoidal constraint for the dual field can be imposed in the path integration (2.8) by means of an auxiliary scalar field λ, which is nothing but a Lagrange multiplier. More concretely, we take

$$Z=\mathcal{N}\int D\hat{\phi}_\alpha D\phi_\alpha D\lambda\,\exp(iS), \tag{2.16}$$

with

$$S=\int_{t_0}^{t_1}dt\Bigg\{\int d^3\vec{x}(L_\alpha(\vec{x},t)+\partial_\alpha\lambda)\hat{\phi}_\beta(\vec{x},t)+i\int d^3\vec{x}d^3\vec{x}'\,\hat{\phi}_\alpha(\vec{x},t)F(|\vec{x}-\vec{x}'|)\hat{\phi}_\alpha(\vec{x}',t)\Bigg\}. \tag{2.17}$$

In Eq. (2.17), the whole dependence on the ϕ fields is implicit in the nonlocal "source" term

$$L_\alpha(\vec{x},t) \equiv -\sum_{\mathcal{P}} \oint_{\partial\mathcal{P}} dx'_\gamma \epsilon_{\alpha\beta\gamma}[\partial_t \phi_\sigma(\vec{x}_p,t)\partial^{-2}\partial_\beta + \phi_\sigma(\vec{x}_p,t)v_\beta(\vec{x}',t) - \nu\phi_\sigma(\vec{x}_p,t)\partial_\beta]\delta^3(\vec{x}'-\vec{x}). \quad (2.18)$$

To proceed, we define the Fourier transform of $L_\alpha(\vec{x},t)$,

$$\tilde{L}_\alpha(\vec{k},t) = \int d^3\vec{x}\exp(-i\vec{k}\cdot\vec{x})L_\alpha(\vec{x},t) = i\sum_{\mathcal{P}}\oint_{\partial\mathcal{P}} dx'_\gamma \epsilon_{\alpha\beta\gamma}\exp(-i\vec{k}\cdot\vec{x}') \times \left(\frac{k_\beta}{k^2}\partial_t\phi_\sigma(\vec{x}_p,t) + i\phi_\sigma(\vec{x}_p,t)v_\beta(\vec{x}',t) + \nu\phi_\sigma(\vec{x}_p,t)k_\beta\right). \quad (2.19)$$

Writing, for a given plaquette $\mathcal{P}=(\vec{x}_p,\hat{x}_\sigma)$, the boundary position vector as $\vec{x}'=\vec{x}_p+\vec{\xi}$, with $v_\alpha(\vec{x}')\simeq v_\alpha(\vec{x}_p)+\xi_\eta\partial_\eta v_\alpha(\vec{x}_p)$, we obtain

$$\tilde{L}_\alpha(\vec{k},t) = i\sum_p \exp(-i\vec{k}\cdot\vec{x}_p)\epsilon_{\alpha\beta\gamma}\left\{g_{\sigma\gamma}(\vec{k})\left(\frac{k_\beta}{k^2}\partial_t\phi_\sigma(\vec{x}_p,t) + i\phi_\sigma(\vec{x}_p,t)v_\beta(\vec{x}_p,t) + \nu k_\beta\phi_\sigma(\vec{x}_p,t)\right) + g_{\sigma\gamma,\eta}(\vec{k})\phi_\sigma(\vec{x}_p,t)\partial_\eta v_\beta(\vec{x}_p,t)\right\}, \quad (2.20)$$

with

$$g_{\sigma\gamma}(\vec{k}) \equiv \oint_{\partial\mathcal{P}} d\xi_\gamma \exp(-i\vec{k}\cdot\vec{\xi}),$$

$$g_{\sigma\gamma,\eta}(\vec{k}) \equiv -i\oint_{\partial\mathcal{P}} d\xi_\gamma\xi_\eta \exp(-i\vec{k}\cdot\vec{\xi}) = \frac{\partial}{\partial k_\eta}g_{\sigma\gamma}(\vec{k}). \quad (2.21)$$

In the limit $\epsilon\to 0$, keeping $k \ll 1/\epsilon$, we have, asymptotically, $g_{\alpha\beta}(\vec{k})=i\epsilon^2\epsilon_{\alpha\beta\gamma}k_\gamma$, and, therefore,

$$\tilde{L}_\alpha(\vec{k},t) = \epsilon^2\sum_p \exp(-i\vec{k}\cdot\vec{x}_p)\{\tilde{\Pi}_{\alpha\beta}(\vec{k})[\partial_t\phi_\beta(\vec{x}_p,t) + \nu k^2\phi_\beta(\vec{x}_p,t)] + i\phi_\beta(\vec{x}_p,t)[\delta_{\alpha\beta}v_\gamma(\vec{x}_p,t)k_\gamma - i\partial_\alpha v_\beta(\vec{x}_p,t) - k_\alpha v_\beta(\vec{x}_p,t)]\}, \quad (2.22)$$

where we used $\tilde{\Pi}_{\alpha\beta}(\vec{k})=\delta_{\alpha\beta}-k_\alpha k_\beta/k^2$, the Fourier-transformed projector on transverse modes. The continuum limit of the above sum is defined through the substitutions

$$\vec{x}_p \to \vec{x},\ \sum_p \to \frac{1}{\epsilon^3}\int d^3\vec{x},\ \phi_\beta \to \epsilon\phi_\beta. \quad (2.23)$$

We find

$$\tilde{L}_\alpha(\vec{k},t) = \tilde{\Pi}_{\alpha\beta}(\vec{k})[\partial_t\tilde{\phi}_\beta(\vec{k},t) + \nu k^2\tilde{\phi}_\beta(\vec{k},t)] + \Omega_\alpha(\vec{k},t), \quad (2.24)$$

where

$$\tilde{\phi}_\alpha(\vec{k},t) = \int d^3\vec{x}\exp(-i\vec{k}\cdot\vec{x})\phi_\alpha(\vec{x},t),$$

$$\Omega_\alpha(\vec{k},t) = \int d^3\vec{x}\exp(-i\vec{k}\cdot\vec{x})v_\beta(\partial_\beta\phi_\alpha - \partial_\alpha\phi_\beta). \quad (2.25)$$

There is a simple connection between ϕ_α and the velocity field v_α. Taking the Fourier transform of the vorticity field, we find

$$\tilde{\omega}_\alpha(\vec{k},t) = \int d^3\vec{x}\exp(-i\vec{k}\cdot\vec{x})\omega_\alpha(\vec{x}) = \sum_{\mathcal{P}}\oint_{\partial\mathcal{P}} dx_\alpha \exp(-i\vec{k}\cdot\vec{x})\phi_\sigma(\vec{x}_p,t) = \sum_p g_{\beta\alpha}(\vec{k})\exp(-i\vec{k}\cdot\vec{x}_p)\phi_\beta(\vec{x}_p,t) = i\epsilon^2\epsilon_{\alpha\beta\gamma}k_\beta\sum_p \exp(-i\vec{k}\cdot\vec{x}_p)\phi_\gamma(\vec{x}_p,t). \quad (2.26)$$

Recalling Eq. (2.23), we get $\omega_\alpha=\epsilon_{\alpha\beta\gamma}\partial_\beta\phi_\gamma$ in the continuum limit. Since $\omega_\alpha=\epsilon_{\alpha\beta\gamma}\partial_\beta v_\gamma$, we immediately conclude that the fields v_α and ϕ_α differ only by a gradient, which means that $v_\alpha=\Pi_{\alpha\beta}\phi_\beta$. The field ϕ_α can be interpreted as a "gauge" velocity field in a model where no gauge fixing has been implemented. The physical gauge-fixing prescription corresponds simply to the imposition of the incompressibility constraint, $\partial_\alpha\phi_\alpha=0$. As a consequence, if Eq. (2.22) is taken back to real space, we get

$$L_\alpha(\vec{x},t) = \partial_t v_\alpha - \nu\partial^2 v_\alpha + v_\beta(\partial_\beta\phi_\alpha - \partial_\alpha\phi_\beta). \quad (2.27)$$

We define at this point the additional scalar field $\zeta=[\partial_t - \nu\partial^2]^{-1}\lambda$, and impose $\phi_\alpha=v_\alpha+\partial_\alpha\zeta$, so that

$$L_\alpha + \partial_\alpha\lambda = \partial_t\phi_\alpha - \nu\partial^2\phi_\alpha + v_\beta(\partial_\beta\phi_\alpha - \partial_\alpha\phi_\beta). \quad (2.28)$$

Since the action (2.17) is quadratic in $\hat{\phi}_\alpha$, it is possible to evaluate the exact path integration over the dual fields. Using Eqs. (2.16), (2.17), and (2.28), the result is an effective (and exact) expression for the probability density functional Z,

$$Z = \mathcal{N}\int D\phi_\alpha D\lambda \exp(iS_\phi), \quad (2.29)$$

where

$$S_\phi = \frac{i}{4}\int_{t_0}^{t_1} dt \int d^3\vec{x}d^3\vec{x}'[\partial_t\phi_\alpha - \nu\partial^2\phi_\alpha + v_\beta(\partial_\beta\phi_\alpha - \partial_\alpha\phi_\beta)]_x \times F^{-1}(\vec{x}-\vec{x}')[\partial_t\phi_\alpha - \nu\partial^2\phi_\alpha + v_\beta(\partial_\beta\phi_\alpha - \partial_\alpha\phi_\beta)]_{x'}. \quad (2.30)$$

The field theory given by Eq. (2.30) may be obtained directly, along the Martin-Siggia-Rose formalism, from the stochastic differential equation

$$\partial_t \phi_\alpha + v_\beta(\partial_\beta \phi_\alpha - \partial_\alpha \phi_\beta) = \nu \partial^2 \phi_\alpha + f_\alpha . \qquad (2.31)$$

For vanishing external forces, the above expression reduces to the usual dipole version of the Navier-Stokes equation [15]. Substituting ϕ_α by $v_\alpha + \partial_\alpha \zeta$ in Eq. (2.31), we get the original stochastic Navier-Stokes Eqs. (2.1), with pressure given by $P = \partial_t \zeta - \nu \partial^2 \zeta - \frac{1}{2}\vec{v}^2$. We have provided, thus, starting from Eq. (2.12), a derivation of Eq. (2.30) [and, equivalently, Eq. (2.31)], which emphasizes the gauge structure associated to the dynamics of the field $\phi_\alpha(\vec{x},t)$. It is tempting to conjecture that some of the standard gauge field theory techniques, like the $1/N$ expansion in non-Abelian extensions, instantons, loop calculus, etc., could find interesting applications in the turbulence domain, an idea formerly advanced by Migdal [28].

Does Eq. (2.31) yield any advantage over the standard Navier-Stokes formulation? Direct numerical simulations based on Eq. (2.31) would probably have the same computational cost than the ones which usually rely on the Navier-Stokes equations, since both versions involve at least two Fourier transformations per iteration cycle. In practice, the above description provides an alternative approach to large eddy simulations [14], or the analysis of phenomenological aspects of vortex tubedynamics, as put forward in the following considerations.

III. STOCHASTIC VORTEX TUBE EVOLUTION

We are now interested in investigating the evolution of a closed vortex tube Γ, with small linear cross-sectional dimensions (of the order of η) and subject to the action of large-scale Gaussian random forces. In a first approximation, we regard the tube as a vorticity filament, parametrized by the curve $x_\alpha = x_\alpha(s,t)$, and carrying total vorticity flux ϕ. The vorticity field is given by

$$\omega_\alpha = \phi \delta(n_1)\delta(n_2)\frac{d}{ds}x_\alpha(s,t), \qquad (3.1)$$

where, similarly to the former plaquette's definitions, n_1 and n_2 indicate the normal and binormal coordinates along the line vortex.

The assumptions taken in Eq. (3.1) that the vorticity flux is time-independent and that cross-section fluctuations may be neglected are imposed as phenomenological constraints. Our results will be expected to hold to the extent that phenomena such as vortex breakdown, vortex merging, etc. do not affect the vortex tube evolution. Such flow regimes have been well verified in the numerical and real experiments where vortices are mostly advected by the background flow, during their mean lifetime, in agreement with the flux conservation Kelvin's theorem. This state of affairs gives in fact the physical basis that supports the somewhat popular choice of modeling vortex tubes by means of Burgers vortices, or similar configurations.

Our first task here is to apply the information provided by Eq. (3.1) in the effective action (2.30). In the limit of vanishing viscosity, we are left, therefore, with the evaluation of $\partial_t \phi_\alpha$ and $v_\beta(\partial_\beta \phi_\alpha - \partial_\alpha \phi_\beta)$. The latter quantity is just minus the Lamb vector. In fact, using $\omega_\alpha = \epsilon_{\alpha\beta\gamma}\partial_\beta \phi_\gamma$, a straightforward computation leads to

$$v_\beta(\partial_\beta \phi_\alpha - \partial_\alpha \phi_\beta) = \epsilon_{\alpha\beta\gamma}\omega_\beta v_\gamma . \qquad (3.2)$$

To find $\partial_t \phi_\alpha$, let us imagine, as an auxiliary construction, that the line vortex is advected by a divergence-free field $\xi_\alpha(\vec{x},t)$, defined on all space, and which satisfies the boundary condition $\xi_\alpha(\vec{x},t) = \dot{x}_\alpha(s,t)$ on Γ. We have, then [29],

$$\partial_t \vec{\omega} = \vec{\nabla} \times (\vec{\xi} \times \vec{\omega}). \qquad (3.3)$$

Observing that $\partial_t \omega_\alpha = \partial_t[\epsilon_{\alpha\beta\gamma}\partial_\beta \phi_\gamma] = \epsilon_{\alpha\beta\gamma}\partial_\beta \partial_t \phi_\gamma$, we get, from Eq. (3.3),

$$\partial_t \phi_\alpha = \epsilon_{\alpha\beta\gamma}\dot{x}_\beta \omega_\gamma + \partial_\alpha \lambda , \qquad (3.4)$$

where λ is an arbitrary field. We are ready to substitute Eqs. (3.2) and (3.4) into Eq. (2.30). Introducing

$$\psi_\alpha^\perp(s,t) \equiv \epsilon_{\alpha\beta\gamma}\psi_\beta(s,t)\frac{d}{ds}x_\gamma(s,t), \qquad (3.5)$$

where $\psi_\alpha = \dot{x}_\alpha - v_\alpha$, we obtain

$$S_\phi = S_{\psi\psi} + S_{\lambda\psi} + S_{\lambda\lambda}, \qquad (3.6)$$

with

$$S_{\psi\psi} = \frac{i\phi^2}{4}\int_{t_0}^{t_1} dt \int_0^{p(t)} ds \int_0^{p(t)} ds' \, \psi_\alpha^\perp(s,t) F^{-1}[\vec{x}(s) - \vec{x}(s')]\psi_\alpha^\perp(s',t),$$

$$S_{\lambda\psi} = \frac{i\phi}{2}\int_{t_0}^{t_1} dt \int d^3\vec{x} \int_0^{p(t)} ds' \, \partial_\alpha \lambda(\vec{x},t) F^{-1}[\vec{x} - \vec{x}(s')]\psi_\alpha^\perp(s',t),$$

$$S_{\lambda\lambda} = \frac{i}{4}\int_{t_0}^{t_1} dt \int d^3\vec{x} \int d^3\vec{x}' \, \partial_\alpha \lambda(\vec{x},t) F^{-1}(\vec{x} - \vec{x}')\partial_\alpha \lambda(\vec{x}',t), \qquad (3.7)$$

where $p(t)$ is the length of the vorticity filament. The integration over λ gives

$$Z = \mathcal{N}\int D\psi_\alpha^\perp \exp(iS_\psi), \qquad (3.8)$$

where

$$S_\psi = \frac{i\phi^2}{4}\int_{t_0}^{t_1} dt \int_0^{p(t)} ds \int_0^{p(t)} ds' \, \psi_\alpha^\perp(s,t)\Pi_{\alpha\beta}F^{-1}[\vec{x}(s) - \vec{x}(s')]\psi_\alpha^\perp(s',t). \qquad (3.9)$$

Note that while S_ψ is a functional of $\psi_\alpha^\perp$, the projection of ψ_α on dx_α/ds (that is, the longitudinal component of ψ_α) maps the line vortex into itself. The singularities that eventually appear in the integrand of Eq. (3.9) may be circumvented in a physical way, replacing the original vortex filament by a vortex tube, through the substitutions

$$\psi_\alpha^\perp(s,t) \to \psi_\alpha^\perp(s,t)h(n_1,n_2),$$

$$\psi_\alpha^\perp(s',t) \to \psi_\alpha^\perp(s',t)h(n_1',n_2'),$$

$$ds \to d^3\vec{x},\ ds' \to d^3\vec{x}',$$

$$F^{-1}[\vec{x}(s) - \vec{x}(s')] \to F^{-1}(\vec{x} - \vec{x}'), \qquad (3.10)$$

where $\vec{x}=(n_1,n_2,s)$, $\vec{x}'=(n_1',n_2',s')$, and

$$h(n_1,n_2) = \frac{1}{\pi\eta^2}\exp\left(-\frac{1}{\eta^2}(n_1^2+n_2^2)\right). \qquad (3.11)$$

We assume that the "smeared" curvature radius of the vortex tube is much larger than the Kolmogorov dissipation length (a hypothesis supported by observations). In practical computations, this allows us to work with a straight vortex tube, taking $s=z$, $n_1=x$, and $n_2=y$ (one can figure it out as a circular vortex tube with infinite curvature radius). Below, we deal with two specific examples of external stochastic forcing, given by Eqs. (2.3) and (2.5), which will be named models A and B, respectively. A more concise expression for S_ψ, compared to Eq. (3.9), follows in general, relying basically on the slender vortex tube profile.

Analysis of model *A*

To evaluate S_ψ, it is necessary to kernel,

$$\Pi_{\alpha\beta}F^{-1}(\vec{x}-\vec{x}') = \frac{1}{8\pi D_0 m}(\delta_{\alpha\beta} - \partial^{-2}\partial_\alpha\partial_\beta) \times(\partial^2 - m^2)^2\delta^3(\vec{x}-\vec{x}'). \qquad (3.12)$$

If this expression is substituted into Eq. (3.9), considering Eqs. (3.10) and (3.11), a number of terms is obtained, hierarchically organized according to the powers of the dissipation length $\eta\to 0$ defined in their coefficients. We will retain in the expression for S_ψ only the dominant term, corresponding to the smallest power of η. Using rotation invariance around the z axis, this prescription effectively amounts to performing in Eq. (3.9) the replacement

$$\Pi_{\alpha\beta}F^{-1}(\vec{x}-\vec{x}') \to \frac{\delta_{\alpha\beta}}{16\pi D_0 m}(\partial_\perp^2)^2\delta^3(\vec{x}-\vec{x}'), \quad (3.13)$$

where $(\partial_\perp)^2 \equiv \partial_x^2+\partial_y^2$. We obtain

$$S_\psi = \frac{i\phi^2}{16\pi^2 D_0 m\eta^6}\int_{t_0}^{t_1}dt\int_0^{p(t)}ds[\psi_\alpha^\perp(s,t)]^2. \quad (3.14)$$

The full expansion in powers of η can be worked out as well, being related to the derivative expansion in powers of ∂_s. We get, in the $m\to 0$ limit,

$$S_\psi = \frac{i\phi^2}{16\pi^2 D_0 m\eta^6}\int_{t_0}^{t_1}dt\int_0^{p(t)}ds[\psi_\alpha^\perp(1 + c_1\eta^2\partial_s^2 + c_2\eta^4\partial_s^4 + \cdots)\psi_\alpha^\perp]. \qquad (3.15)$$

It is not necessary to write down the explicit values of the c_i's, insofar as they will not have any relevant role in the forthcoming arguments. Although Eq. (3.14) is an apparently elementary quadratic action, the time-dependent spatial integration limit $p(t)$ renders the analytical evaluation of Z difficult. Nevertheless, the problem looks amenable of numerical investigation through the use of Langevin techniques [30].

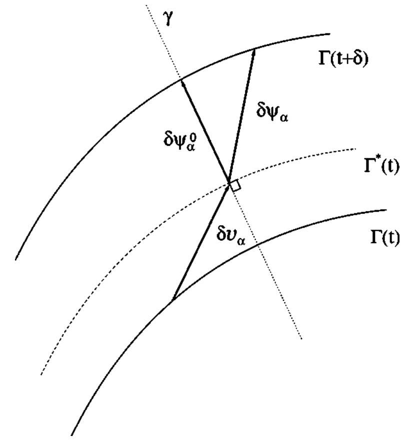

FIG. 2. The vortex tube $\Gamma(t)$ evolves, during the small time interval δ, to the new configuration $\Gamma(t+\delta)$. The intermediate dashed tube $\Gamma^*(t)$ corresponds to the transport provided by the self-induced velocity field v_α.

Since we are discussing the time evolution of a vortex tube, the relevant physical question one may ask is concerned with the probability density functional of finding the tube in a certain geometrical configuration. In a first instance, this seems to be an intricate problem, once any individual vortex tube "world line" to be considered in the path integration is accounted for by a large number of configurations of ψ_α. A simple solution of this degeneracy problem may be obtained, however, by means of the "minimal mapping" ψ_α^0, depicted in Fig. 2. The essential idea is to keep track of the vortex tube evolution for a very small time interval δ. We decompose the time evolution in two steps. First, the tube $\Gamma(t)$ is mapped into $\Gamma^*(t)$ through its self-induced velocity field v_α. Next, the stochastic perturbation ψ_α takes $\Gamma^*(t)$ to the final configuration $\Gamma(t+\delta)$. The mapping sequence $x_\alpha \to x_\alpha' \to x_\alpha''$, with

$$x_\alpha' = x_\alpha + \delta v_\alpha,$$

$$x_\alpha'' = x_\alpha' + \delta\psi_\alpha. \qquad (3.16)$$

Let γ be the plane that contains x_α' and is normal to $\Gamma^*(t)$. Then, ψ_α^0 is just the vector parallel to γ that connects x_α' to the vortex tube $\Gamma(t+\delta)$. We have

$$\psi_\alpha^\perp(s,t) = \psi_\alpha^0(s+\delta\psi_s(s,t),t) + O(\delta^2) = \psi_\alpha^0(s,t) + \delta\psi_s\partial_s\psi_\alpha^0(s,t) + O(\delta^2), \qquad (3.17)$$

where $\psi_s \equiv \psi_\alpha dx_\alpha/ds$. The expansion (3.17) implies that $\psi_\alpha^\perp - \psi_\alpha^0 = O(\delta)$, and so $\psi_\alpha^\perp$ may be substituted by ψ_α^0 in Eq. (3.14). We find, thus, that the probability density functional for the transition $\Gamma(t_0)\to\Gamma(t_1)$ of the vortex tube configuration may be defined as

$$Z_A = \mathcal{N} \int D\psi_\alpha^0 \times \exp\left\{ - \frac{\phi^2}{16\pi^2 D_0 m \eta^6} \int_{t_0}^{t_1} dt \int_0^{p(t)} ds [\psi_\alpha^0(s,t)]^2 \right\}. \tag{3.18}$$

Clearly, the original field degeneracy is removed, and there is in Eq. (3.18) a one-to-one correspondence between the vortex tube integration paths and the fields $\psi_\alpha^0(s,t)$.

Analysis of model *B*

The computational steps are exactly the same as the ones performed in the former case. The only technical difference is that the analog of Eq. (3.12) is written now in Fourier space,

$$\tilde{\Pi}_{\alpha\beta}\tilde{F}^{-1}(k) = \frac{1}{2\pi^2 D_0}(\delta_{\alpha\beta}k^3 - k_\alpha k_\beta k). \tag{3.19}$$

The dominant contribution to Eq. (3.9), of order $1/\eta^5$, comes from the substitution

$$\tilde{\Pi}_{\alpha\beta}\tilde{F}^{-1}(k) \to \frac{1}{4\pi^2 D_0}\delta_{\alpha\beta}k_\perp^3. \tag{3.20}$$

We get, similarly to Eq. (3.18), the probability density functional

$$Z_B = \mathcal{N} \int D\psi_\alpha^0 \exp\left\{ - \frac{6\phi^2\sqrt{\pi}}{D_0 \eta^5} \int_{t_0}^{t_1} dt \int_0^{p(t)} ds [\psi_\alpha^0(s,t)]^2 \right\}. \tag{3.21}$$

A remarkable feature of model *B*, as it may be easily inferred from Eq. (3.21), is that there is no dependence of the probability density functional Z_B upon the integral scale $L=1/m$ (as it occurs in model *A*, for instance).

In order to establish a connection between the above models and observed features of turbulent flows, a slight modification of expressions (3.18) and (3.21) is necessary. In principle, the Martin-Siggia-Rose framework implemented by Eqs. (2.29) and (2.30) is expected to provide a bona fide statistical modeling of vortex tube motion if an ultraviolet cutoff appears dynamically at a frequency $|\omega| \sim 1/t_\eta$, where $t_\eta \sim \eta^{2/3}$ is the eddy turnover time at the Kolmogorov length scale. The simplest way to find improved versions of Eqs. (3.18) and (3.21), thus, is to replace the Dirac δ factor in Eq. (2.2) by a regularized expression like

$$\delta_R(t-t') = \frac{1}{2t_\eta}\exp(-t_\eta^{-1}|t-t'|), \tag{3.22}$$

and relax the cutoff prescription for the field ψ_α in frequency space. As a consequence, if all the steps leading to Eqs. (3.18) and (3.21) are evaluated again, taking into account the modifications due to Eq. (3.22), we will get, for both models *A* and *B*, the general result

$$Z = \mathcal{N} \int D\psi_\alpha^0 \exp\left\{ - c \int_{t_0}^{t_1} dt \int_0^{p(t)} ds [(t_\eta \partial_t \psi_\alpha^0)^2 + (\psi_\alpha^0)^2] \right\}, \tag{3.23}$$

where $c=c(m,\eta,D_0)$ takes, for models *A* and *B*, the same values as before.

IV. BACKGROUND VELOCITY FLUCTUATIONS

It is interesting to note that the probability density functional (3.23) is completely equivalent to the one derived for the problem of random advection of a vortex tube by a background velocity field. In this way, we can draw a correspondence between the former effective description, based on the analysis of the stochastic Navier-Stokes equations, and realistic properties of turbulent flows. Such a mapping, however, is not unique: it turns out that there is an infinity of velocity-velocity correlators that would work. Having this theoretical limitation in mind, we may try, at best, to compare the form of the predicted coupling constants appearing in Eq. (3.23), with the ones found from experimental and numerical studies. The central problem, then, is to define a Gaussian stochastic forcing which leads, in an accurate way, to known features of the background flow. At this point we are guided by numerical observations [9], which indicate the existence of a short-range correlated background flow.

If $v_\alpha(\vec{x},t)$ is the velocity of the background flow, which is assumed to be a random Gaussian fluctuating field with vanishing mean value, a particularly appealing correlator is defined as

$$\langle v_\alpha(\vec{x},t) v_\beta(\vec{x}',t') \rangle = g\Pi_{\alpha\beta}\delta^3(\vec{x}-\vec{x}')\delta_R(t-t'). \tag{4.1}$$

It follows that the one-dimensional background energy spectrum is given by

$$E(k) = \frac{gk^2}{4\pi^2 t_\eta}, \tag{4.2}$$

and that the path-integral expression (3.23) holds, with

$$c = \frac{\pi\eta^2}{2g}. \tag{4.3}$$

It is straightforward to prove Eq. (4.2) from the Fourier transform of the velocity-velocity correlator (4.1). Let us discuss now, in more detail, how Eq. (3.23) arises from Eq. (4.1), with the specific parameter definition (4.3).

The probability density functional to have a certain background velocity field $\overline{v_\alpha}(\vec{x},t)$ in the region Ω_t enclosed by a vortex tube, for the time interval $t_0 \leq t \leq t_1$, may be written as

$$\mathcal{P} = \langle \Pi_{i,j}\delta(\overline{v_\alpha}(\vec{x}_i,t_j) - v_\alpha(\vec{x}_i,t_j)) \rangle, \tag{4.4}$$

where $(\vec{x}_i,t_j)$ denotes a discretized space-time position defined in the set of world lines generated by the vortex tube evolution. Using the Fourier representation of the δ function, Eq. (4.4) becomes, in the continuum limit,

$$\mathcal{P} = \mathcal{N} \int D\xi_\alpha \exp\left(i \int_{t_0}^{t_1} dt \int_{\Omega_t} d^3\vec{x} \xi_\alpha \overline{v_\alpha} \right) \times \left\langle \exp\left(- i \int_{t_0}^{t_1} dt \int_{\Omega_t} d^3\vec{x} \xi_\alpha v_\alpha \right) \right\rangle. \quad (4.5)$$

Resorting to the Gaussian random behavior of the background velocity field, we are able to compute the above expectation value. Using Eq. (4.1), we find

$$\mathcal{P} = \mathcal{N} \int D\xi_\alpha \exp\left(i \int_{t_0}^{t_1} dt \int_{\Omega_t} d^3\vec{x} \xi_\alpha \overline{v_\alpha} \right) \times \exp\left[- \frac{g}{2} \int_{t_0}^{t_1} dt \int_{t_0}^{t_1} dt' \times \int_{\Omega_t} d^3\vec{x} \xi_\alpha(\vec{x},t) \delta_R(t-t') \Pi_{\alpha\beta} \xi_\beta(\vec{x},t') \right]. \quad (4.6)$$

Since $\overline{v_\alpha} = \Pi_{\alpha\beta} \overline{v_\beta}$, we may integrate over the field ξ_α to get

$$\mathcal{P} \propto \exp\left\{ - \frac{1}{2g} \int_{t_0}^{t_1} dt \int_{\Omega_t} d^3\vec{x} [(t_\eta \partial_t \overline{v_\alpha})^2 + (\overline{v_\alpha})^2] \right\}. \quad (4.7)$$

If the vortex tube has a small circular cross section of area $\pi\eta^2$, we can replace $\int_{\Omega_t} d^3\vec{x}$ by $\pi\eta^2 \int_0^{p(t)} ds$ in Eq. (4.7). Furthermore, to find the transition probability density functional Z for the vortex tube evolution between configurations $\Gamma(t_0)$ and $\Gamma(t_1)$, we (i) decompose the velocity field in transverse and longitudinal components to the vortex tube tangent vector, viz., $\overline{v_\alpha} = \overline{v_\alpha^\perp} + \overline{v_\alpha^\dagger}$, (ii) integrate over the longitudinal components $\overline{v_\alpha^\dagger}$, and (iii) introduce the "minimal velocity field" v_α^0 in close analogy with the previous definition of ψ_α^0. We obtain

$$Z = \mathcal{N} \int Dv_\alpha^0 \exp\left\{ - \frac{\pi\eta^2}{2g} \int_{t_0}^{t_1} dt \int_0^{p(t)} ds [(t_\eta \partial_t v_\alpha^0)^2 + (v_\alpha^0)^2] \right\}. \quad (4.8)$$

Therefore, identifying v_α^0 to ψ_α^0, we have just found Eq. (3.23) again, with c given by Eq. (4.3).

We remark that it is possible to define, without much additional effort, an alternative form for the velocity-velocity correlator which would lead to the expanded formulation given by Eq. (3.15). We could take, for instance,

$$\langle v_\alpha(\vec{x},t) v_\beta(\vec{x}',t') \rangle = g \Pi_{\alpha\beta} f(x-x') f(y-y') f(z-z') \delta_R(t-t'), \quad (4.9)$$

where x, y, and z are local coordinates attached to the vortex tube, and

$$f(x) \sim \int dk \frac{1}{1 + b_1 \eta^2 k^2 + b_2 \eta^4 k^4 + \cdots} \exp(ikx). \quad (4.10)$$

The coefficients b_i may be adjusted in order to recover the set of c_i's appearing in Eq. (3.15). Of course, Eq. (4.9) is approximately isotropic in the inertial range wave numbers, where $k \ll 1/\eta$. In any case, however, Eq. (4.9) should yield an isotropic correlator, when written in terms of the coordinates (x,y,z) of a fixed Cartesian framework, after the average over all possible vortex tube orientations is taken.

From Eqs. (4.2) and (4.3), we can predict the form of the one-dimensional energy spectrum for models A and B (disregarding numerical prefactors),

$$E_A(k) \sim \frac{D_0 m \eta^8}{\phi^2 t_\eta} k^2,$$

$$E_B(k) \sim \frac{D_0 \eta^7}{\phi^2 t_\eta} k^2. \quad (4.11)$$

It is useful to compare Kolmogorov's spectrum $E_K(k) \sim D_0^{2/3} k^{-5/3}$ with the above expressions. We may estimate, relying on Kolmogorov phenomenology, that $\phi \sim D_0^{1/3} \eta^{4/3}$ and $t_\eta \sim D_0^{-1/3} \eta^{2/3}$. At the dissipative wave number $k_\eta \sim 1/\eta$, we define the Reynolds-number-dependent dimensionless ratio

$$Q \equiv \frac{E(k_\eta)}{E_K(k_\eta)} \sim R_e^\alpha, \quad (4.12)$$

where $E(k_\eta)$ is the background spectrum for a given model. It turns that for model A, we get $\alpha = -1$ while for model B, $\alpha = 0$. More generally, it is not difficult to realize that the family of Gaussian stochastic forces described by

$$\tilde{F}(k) \sim (k^2 + m^2)^{-\beta}, \quad (4.13)$$

with $\beta \geq \frac{3}{2}$ leads to Eq. (4.12) with $\alpha = 3 - 2\beta$.

A numerical wavelet analysis by Farge *et al.* [9] of the direct numerical simulations carried out by Vincent and Meneguzzi [8] at moderately high Reynolds numbers reveals the existence of a background k^2 one-dimensional energy spectrum. The turbulent flow may be depicted as a vortex tube gas surrounded by incoherent fluctuations, the latter having their kinetic energy equiprobably distributed over the spatial Fourier modes. It has been suggested in Ref. [9] that the dissipation at the bottom of the inertial range would be preceded at larger scales by some coherent-to-incoherent energy transfer from the vortex tubes to the background field. A fraction of the vortex tubes would be disrupted in a conservative way, so that the transformation of their mechanical energy into heat would occur afterwards in the background flow. One may conjecture that the integral length scale is irrelevant in this sequence of small-scale events. In that case, we have $\alpha = 0$, as in model B, which is actually the scenario indicated by the numerical results, where $Q \simeq 0.1$ for the Taylor-scale Reynolds number $R_\lambda = 150$ (equivalent to $R_e \simeq 10^3$, according to Ref. [8] and also using the phenomenological expressions of Lohse [31]).

V. CONCLUSION

We investigated in this work both formal and phenomenological aspects of homogeneous isotropic turbulence, within the stochastic modeling of vorticity dynamics. A rigorous statistical lattice vortex description of turbulent flows was established, yielding the basis for a subsequent phenomenological discussion of the problem of the random evolution of vortex tubes, commonly observed in experiments and numerical simulations. Since the advection of vorticity coherent structures is ultimately caused by the background flow, according to Kelvin's theorem, we interpret the stochastic method as an effective tool for computing the evolution of vortex tubes. We were able to find in this way a plausible form for the background velocity-velocity correlator, and, as an immediate consequence, the background one-dimensional energy spectrum. We found a satisfactory agreement with the recent numerical analysis of Farge *et al.* [9], where a thermal-like spectrum was clearly noticed for the background flow. In particular, we observed that the Gaussian correlator (2.5), used in the renormalization-group approach to turbulence [19–22]—which has led to perhaps the best theoretical computation of the Kolmogorov spectrum performed so far—is likely the correct choice (model B of Sec. IV) for the derivation of phenomenologically meaningful results. It would be important to improve the connection between the stochastic modeling and the numerical results, concerning anisotropic effects, as the reported zero helicity distribution peak for the incoherent fluctuations [9].

There is strong numerical evidence that the vortex tube gas accounts on its own for Kolmogorov's spectrum [9,16,32,33]. Regarding the background flow, our analysis suggests that it has a twofold character, involving the combination of the "eddy noise" [24] forcing, effectively modeled by Eq. (2.5), and of configurations which satisfy the energy equipartion principle. The picture that emerges—to be explored in further analytical and numerical works—is that these two facets of the background fluctuations are self-consistently related to the vorticity coherent structures. While the force-force correlator (4.13) with $\beta > 3/2$ is a reasonable choice for a rigorous study of the turbulence problem, it becomes useless when considered in the simplified phenomenological perspective addressed in Sec. IV. On the other hand, model B is favored by the force of numerical observations, since it copes well with the tripartite phenomenological stage set up by the vortex tube gas, stochastic eddy noise, and the thermal-like background flow.

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